Multiple Choice

1 (A) $4-x>0, \therefore x<4$

2 (D) $(2x)^2 = x(x+9)$

3 (C) $\dfrac{\frac{1}{2}}{1+\frac{x^2}{4}}$

4 (B) Guide graph is $y = x^2(x^2-1)$

5 (A) $x = 2\cos \pi t$ has period 2, amplitude 2

6 (B) $\sin 2x = 2\sin x \cos x = 2\times\frac{1}{4}\times -\frac{\sqrt{15}}{4}$ (2nd quad, $\cos x<0$)

7 (C) Product of roots $= -1 \times \alpha\beta = -1, \therefore \beta = \frac{1}{\alpha}$

8 (A) 6 letters including 2 Ps and 3 Ls together which count as 1 letter.

9 (D) $\cos^{-1}(-\sin x) = \pi - \cos^{-1}(\sin x)$

$$= \pi - \left(\frac{\pi}{2} - x\right)$$

$$= \frac{\pi}{2} + x, \text{ for } -\frac{\pi}{2} \le x \le \frac{\pi}{2}.$$

10 (B) Only at $(-1,-1)$ and a point where $x>0, y>0$ as the curve $f(x) = -\sqrt{1+\sqrt{1+x}}$ starts at $(-1,-1)$ and is monotonic decreasing, $\therefore f^{-1}(x)$ starts at $(-1,-1)$ and is monotonic increasing.

Question 11

(a) $m_1 = 3, m_2 = \frac{1}{2}, \therefore \tan\theta = \dfrac{3-\frac{1}{2}}{1+\frac{3}{2}} = 1, \therefore \theta = \frac{\pi}{4}.$

(b) $\dfrac{x}{x+1} < 2$

$x(x+1) < 2(x+1)^2$

$(x+1)(2x+2-x) > 0$

$(x+1)(x+2) > 0$

$x < -2$ or $x > -1$

(c) $\angle ACB = 90°$ (semi-circle angle)

$\angle CAB = \angle BCD$ (angles in alternate segments)

$\therefore \Delta ABC \,|||\, \Delta BCD$ (AA)

$\therefore \angle ABC = \angle CBD$ (corresponding angles in similar Δs)

(d) By long division,

$x^3 + 2x^2 - 3x - 7 = (x-2)(x^2+4x+5)+3$

$\therefore Q(x) = x^2 + 4x + 5.$

(e) $\int 2\sin^2 4x dx = \int (1-\cos 8x)dx = x - \frac{\sin 8x}{8} + C.$

(f) (i) $0.95^8 \simeq 0.66$

(ii) $\Pr(\text{at least } 2) = 1 - \Pr(x=0,1)$

$= 1 - 0.95^8 - {}^8C_1(0.05)(0.95)^7$

$\simeq 0.057.$

Question 12

(a) $\frac{dA}{dt}=\frac{dA}{dB}\frac{dB}{dt}=-\frac{9}{B^2}\times 0.2=-\frac{1.8}{\left(\frac{9}{12}\right)^2}=-3.2\text{ ms}^{-1}$

(b) (i) $x=2\sqrt{2}\cos\left(3t-\frac{\pi}{3}\right)$

(ii) $x=2\sqrt{2},-2\sqrt{2}.$

(iii) $\dot{x}=-6\sqrt{2}\sin\left(3t-\frac{\pi}{3}\right)$

When $\sin\left(3t-\frac{\pi}{3}\right)=-\frac{1}{2},$

$3t-\frac{\pi}{3}=-\frac{\pi}{6}$

$3t=\frac{\pi}{6}$

$t=\frac{\pi}{18}$ is the first time.

(c) The tangent at P is $y=px-ap^2$.

Substituting $x=0$ gives $y=-ap^2,\therefore R\left(0,-ap^2\right)$.

$SR=a-(-ap^2)=a+ap^2$.

$SP=PM$ (by definition)

$=ap^2-(-a)=ap^2+a$.

$\therefore SR=SP,$

$\therefore \Delta SRP$ is isosceles.

$\therefore \angle SPR=\angle SRP.$

(d) (i) $T=3+Ae^{kt},\therefore\frac{dT}{dt}=kAe^{kt}=k(T-3).$

(ii) Sub. $t=0,T=30$ gives $30=3+A,\therefore A=27.$

Sub. $t=15,T=28$ gives $28=3+27e^{15k},\therefore k=\frac{\ln\frac{25}{27}}{15}$

Sub. $t=60,T=3+27e^{4\ln\frac{25}{27}}=22.8°$

Question 13

(a) $u=\cos^2 x, du=-2\cos x\sin x dx=-\sin 2x dx.$

When $x=0,u=1$. When $x=\frac{\pi}{4},u=\frac{1}{2}.$

$\int_0^{\frac{\pi}{4}}\frac{\sin 2x}{4+\cos^2 x}dx=\int_1^{\frac{1}{2}}\frac{-du}{4+u}=\left[\ln(4+u)\right]_{\frac{1}{2}}^{1}=\ln\frac{10}{9}.$

(b) ${}^{20}C_k 5^k 2^{20-k}={}^{20}C_{k+1}5^{k+1}2^{19-k}.$

$\frac{20!\times 2}{k!(20-k)!}=\frac{20!\times 5}{(k+1)!(19-k)!}$

$2(k+1)=5(20-k)$

$7k=98$

$k=14$

(c) (i) $a=\frac{d}{dx}\left(\frac{1}{2}v^2\right)=-2e^{-x}.$

$\frac{1}{2}v^2=2e^{-x}+C$

Sub. $x=0,v=2$ gives $C=0.$

$\therefore v^2=4e^{-x}.$

$\therefore v=2e^{-\frac{x}{2}}$, noting initially $v>0$.

(ii) $\frac{dx}{dt}=2e^{-\frac{x}{2}}$

$\int_0^x e^{\frac{x}{2}}dx=2\int_0^t dt$

$2\left(e^{\frac{x}{2}}-1\right)=2t.$

$e^{\frac{x}{2}}=t+1.$

$\therefore x=2\ln(t+1)$

(d) (i) At A, $y=-x.$

$18\sqrt{3}t-5t^2=-18t.$

$\therefore t=\frac{18\left(\sqrt{3}+1\right)}{5}.$

$\therefore OA=\sqrt{2}x=18\sqrt{2}t=\frac{324\sqrt{2}\left(\sqrt{3}+1\right)}{5}$

(ii) $\dot{x}=18,$

$\dot{y}=18\sqrt{3}-10t=18\sqrt{3}-2\times 18\left(\sqrt{3}+1\right)=-18\left(\sqrt{3}+2\right)$

$\tan\alpha=\frac{-18\left(\sqrt{3}+2\right)}{18}=-\left(\sqrt{3}+2\right).$

$\therefore \alpha=-75°.$

$\therefore$ It makes an angle of 30° with the sloping plane.

Question 14

(a) Let $n=1$, LHS $=1(1!)=1$, RHS $=2!-1=1$.

$\therefore$ true when $n=1$.

Assume $1(1!)+2(2!)+...+n(n!)=(n+1)!-1$.

RTP $1(1!)+2(2!)+...+(n+1)(n+1)!=(n+2)!-1$.

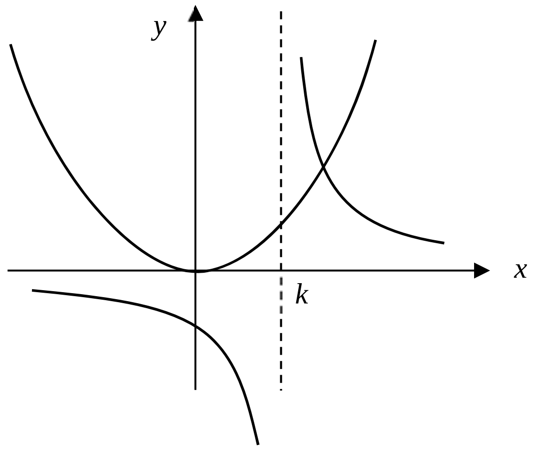
LHS $=(n+1)!-1+(n+1)(n+1)!$

$\quad=(n+1)!(n+1+1)-1$

$\quad=(n+2)!-1=$ RHS.

$\therefore$ True for all $n\ge 1$ by the principle of Math Induction.

(b) (i)



As seen from the diagram, the 2 curves meets only once,

$\therefore x^2=\dfrac{1}{x-k}$ has only one real zero.

$\therefore x^3-kx^2-1=0$ has only one real zero.

(ii) Let $f(x)=x^3-kx^2-1$.

$f'(x)=3x^2-2kx.$

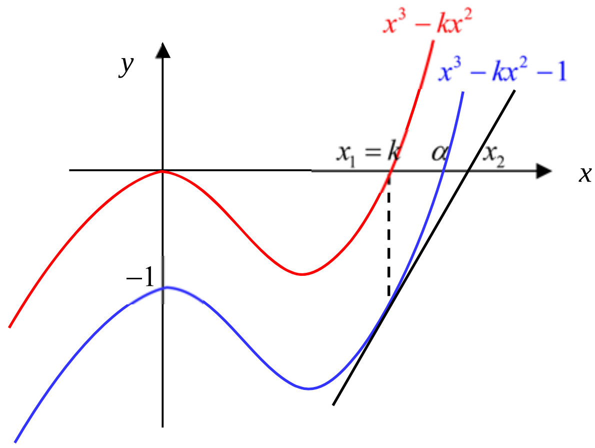
$\therefore f(k)=k^3-k^3-1=-1$

and $f'(k)=3k^2-2k^2=k^2$.

$x_2=k-\dfrac{f(k)}{f'(k)}$

$\quad=k+\dfrac{1}{k^2}.$

(iii) Consider the graph of $f(x)=x^3-kx^2-1$.



By Newton's method, the next approximation is found when the tangent to the curve at the previous approximation meets the x-axis.
From the diagram, as the curve is concave up at the neighbourhood of $x=\alpha$, $x_1<\alpha<x_2$.

(c) (i) As the 2 curves have a common tangent at x_0, they have same gradient at $x=x_0$.

$\therefore \cos x_0=\cos(x_0-\alpha)$.

(ii) Solving $\cos x_0=\cos(x_0-\alpha)$ gives

$x_0=-x_0+\alpha$ (ignore $+2k\pi$, as $x_0<\dfrac{\pi}{2}$. Also ignore $x_0-\alpha$ as this is absurd)

$\therefore \sin x_0=\sin(-x_0+\alpha)$

$\quad=-\sin(x_0-\alpha)$.

(iii) From $x_0=-x_0+\alpha$ we have $x_0=\dfrac{\alpha}{2}$.

$k+\sin(x_0-\alpha)=\sin x_0$.

$k=\sin x_0-\sin(x_0-\alpha)$

$\quad=2\sin x_0$

$\quad=2\sin\dfrac{\alpha}{2}$.